Problem 1 (40 points)

Consider the syntax of the language $E$ for the “palm calculator”.

\[
E \rightarrow n \quad \text{number} \\
| \quad x \quad \text{variable} \\
| \quad E + E \quad \text{addition} \\
| \quad E - E \quad \text{subtraction} \\
| \quad (E) \quad \text{parenthesis} \\
| \quad \text{let } x = E \text{ in } E \quad \text{binding}
\]

We will implement this language $E$ with $<$S,E,C>-machine. The $<$S,E,C>-machine is an abstract machine. $S$ is a stack of values (ordered sequence). $E$ is an environment (function : Variable $\rightarrow$ Value). $C$ is a command sequence defined as following:

\[
C \rightarrow \text{add}$-C /* add top two values and push the result back */ \\
| \quad \text{sub}$-C \\
| \quad \text{bind}(x)$-C /* bind $x$ in $E$ with the value of stack-top, and pop */ \\
| \quad \text{unbind}(x)$-C /* delete the most-recent binding $x$ in $E$ */ \\
| \quad \text{push}(x)$-C /* push the value of the variable $x$ */ \\
| \quad \text{push}(n)$-C /* push the constant $n$ */ \\
| \quad \varepsilon \quad ; \text{empty command}
\]

(a) Define the denotational semantics of the language $E$
(b) Define the transition semantics (abstract machine semantics) of the command $C$.
   If you need a new command, define its syntax and transition semantics freely.
   You may use your own function if it is “well-defined”.
(c) Define the compilation rules $\triangleright$ from the program $E$ to the command $C$ sequence
    with inference rules

Problem 2 (40 points)

Let $P$ be the vertical domain of the natural numbers, \{0, 1, 2, ..., $\infty$\} where

\[0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq 3 \sqsubseteq \ldots \sqsubseteq \infty,\]

and $P'$ the two-element domain \{$\bot$, $\top$\} where $\bot \sqsubseteq \top$.

Then the monotone function $f(x) \equiv \text{if } x = \infty \text{ then } \top \text{ else } \bot$ is not continuous.
(a) Why is the function \( f \) “not continuous”? Explain.

(b) Let a function \( f_k \) be \( f_k(x) \equiv \text{if } x \leq k \text{ then } \bot \text{ else } T \). Then, is the function \( f_k \) continuous? Explain why or why not?

(c) Does a sequence \( f_0 \ f_1 \ f_2 \ldots \) constitute a chain? Why or why not? What is the least element among \( f_k \) where \( k = 0, 1, \ldots, \infty \)?

\[ (f \sqsubseteq g \iff \forall x \in P. f(x) \sqsubseteq g(x)) \]

**Problem 3 (20 points)**

The set \( T \) is defined to consist only of \( t \) that are inductively defined as follows:

\[ t \rightarrow \cdot \mid / t, t/ \]

Every \( t \in T \) enjoys several invariant conditions on the numbers of commas and slashes in \( t \). Prove by induction that the number of slashes in \( t \) is \((2 \times \text{the number of commas})\) for any element of \( T \). First, state your theorem formally (with some mathematical notation), then prove your theorem by induction.