Consider the syntax of the language $E$ for the “palm calculator”. 

$$
E \to n \quad \text{number} \\
| \quad x \quad \text{variable} \\
| \quad E + E \quad \text{addition} \\
| \quad E - E \quad \text{subtraction} \\
| \quad (E) \quad \text{parenthesis} \\
| \quad \text{let } x = E \text{ in } E \quad \text{binding}
$$

We will implement this language $E$ with $<S,E,C>$-machine. The $<S,E,C>$-machine is an abstract machine. $S$ is a stack of values (ordered sequence). $E$ is an environment (function $: \text{Variable} \to \text{Value}$). $C$ is a command sequence defined as following:

$$
C \to \begin{align*}
\text{add·C} & \quad /* \text{add top two values and push the result back */} \\
\text{sub·C} & \\
\text{bind(x)·C} & \quad /* \text{bind x in E with the value of stack-top, and pop */} \\
\text{unbind(x)·C} & \quad /* \text{delete the most-recent binding x in E */} \\
\text{push(x)·C} & \quad /* \text{push the value of the variable x */} \\
\text{push(n)·C} & \quad /* \text{push the constant n */} \\
\varepsilon & \quad ; \text{empty command}
\end{align*}
$$

(a) Define the denotational semantics of the language $E$

(b) Define the transition semantics (abstract machine semantics) of the command $C$.

If you need a new command, define its syntax and transition semantics freely.

You may use your own function if it is “well-defined”.

(c) Define the compilation rules $\triangleright$ from the program $E$ to the command $C$ sequence with inference rules

Problem 2 (40 points)

Let $\mathbf{P}$ be the vertical domain of the natural numbers, $\{0, 1, 2, \ldots, \infty\}$ where

$$
0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq 3 \sqsubseteq \ldots \sqsubseteq \infty,
$$

and $\mathbf{P}'$ the two-element domain $\{\bot, \top\}$ where $\bot \sqsubseteq \top$.

Then the monotone function $f \; x \equiv \text{if } x = \infty \text{ then } \top \text{ else } \bot$ is not continuous.
(a) Why is the function $f$ “not continuous”? Explain.
(b) Let a function $f_k$ be $f_k(x) \equiv \{ \begin{array}{ll} \bot & \text{if } x \geq k \\ \top & \text{else} \end{array}$. Then, is the function $f_k$ continuous? Explain why or why not?
(c) Does a sequence $f_0, f_1, f_2, \ldots$ constitute a chain? Why or why not? What is the least element among $f_k$ where $k = 0, 1, \ldots, \infty$?

Problem 3 (20 points)
Set $D \equiv \mathbb{N} \times \mathbb{N}$ consists only of the elements $(n, m)$ that are inductively defined as follows:

- $(n, n) \in D$
- $(n, m) \in D \implies (n, n \times m) \in D$

Prove that

For any $(n, m) \in D$, there exists $i \in \mathbb{N}$ such that $m = n^i$. 