CS520 Theory of Programming Languages

(Theory Area)
Phd. Qualifying Exam (January 13, 2017)

**Problem 1 (40 points)**

Consider the syntax of the language $E$ for the “palm calculator”.

$$E \rightarrow n \quad \text{number}$$
$$| \quad x \quad \text{variable}$$
$$| \quad E + E \quad \text{addition}$$
$$| \quad E - E \quad \text{subtraction}$$
$$| \quad (E) \quad \text{parenthesis}$$
$$| \quad \text{let } x = E \text{ in } E \quad \text{binding}$$

We will implement this language $E$ with $\langle S, E, C \rangle$-machine. The $\langle S, E, C \rangle$-machine is an abstract machine. $S$ is a stack of values (ordered sequence). $E$ is an environment (function : Variable $\rightarrow$ Value). $C$ is a command sequence defined as following:

$$C \rightarrow \text{add} \cdot C$$
$$| \quad \text{sub} \cdot C$$
$$| \quad \text{bind}(x) \cdot C$$
$$| \quad \text{unbind}(x) \cdot C$$
$$| \quad \text{push}(x) \cdot C$$
$$| \quad \text{push}(n) \cdot C$$
$$| \quad e \quad ; \text{empty command}$$

(a) Define the denotational semantics of the language $E$
(b) Define the transition semantics (abstract machine semantics) of the command $C$.
(c) Define the compilation rules $\triangleright$ from the program $E$ to the command $C$ sequence with inference rules

**Problem 2 (40 points)**

Let $P$ be the vertical domain of the natural numbers, $\{0, 1, 2, \ldots, \infty\}$ where

$$0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq 3 \sqsubseteq \ldots \sqsubseteq \infty,$$

and $P'$ the two-element domain $\{\bot, \top\}$ where $\bot \sqsubseteq \top$.

Then the monotone function $f x \equiv \text{if } x = \infty \text{ then } \top \text{ else } \bot$ is not continuous.

(a) Why is the function $f$ “not continuous”? Explain.
(b) Let a function $f_k$ be $f_k x \equiv \text{if } x \leq k \text{ then } \bot \text{ else } \top$. Then, is the function $f_k$
continuous? Explain why or why not?

(c) Does a sequence \( f_0, f_1, f_2, \ldots \) constitute a chain? Why or why not? What is the least element among \( f_k \) where \( k = 0, 1, \ldots, \infty \)?

\[ f \sqsubseteq g \text{ iff } \forall x \in P. f(x) \sqsubseteq g(x) \]

**Problem 3 (20 points)**

Set \( D \subseteq \mathbb{N} \times \mathbb{N} \) consists only of the elements \((n, m)\) that are inductively defined as follows:

\[
\begin{align*}
(n, n) & \in D \\
(n, m) \in D & \quad (n, n \times m) \in D
\end{align*}
\]

Prove that

*For any \((n, m) \in D\), there exists \( i \in \mathbb{N} \) such that \( m = n^i \).*