Qualifying Exam for Algorithm. Jan., 2015.

1. (10 pts) If P = NP, is NP = co-NP? Justify your answer.

2. (5 pts x 4 = 20 pts) Given 2 decision problems L_1 and L_2 in NP and $L_2 \leq_p L_1$ (i.e., L_2 is polynomial-time reducible to L_1), for each of the following statements, give one of T(true), F(false), or O(open question), and briefly justify your answer. (NPC denotes the set of NP-complete problems.)
   (1) If $L_1 \in$ P, then $L_2 \in$ P.
   (2) If $L_2 \in$ P, then $L_1 \in$ P.
   (3) If $L_1 \in$ NPC, then $L_2 \in$ NPC.
   (4) If $L_2 \in$ NPC, then $L_1 \in$ NPC.

3. (20 pts) Give a dynamic programming algorithm for finding all-pairs shortest paths of a directed weighted graph and analyze your algorithm. (A more efficient algorithm will earn more points.)

4. Consider the following divide-and-conquer algorithm to multiply two n-bit numbers X and Y. We divide X into A and B and Y into C and D, where A, B, C, D are n/2-bit numbers.

$$X = 2^{n/2}A + B$$

A	B

$$Y = 2^{n/2}C + D$$

C	D

$$XY = 2^n AC + 2^{n/2}BC + 2^{n/2}AD + BD$$

(a) (10 pts) Give a recurrence for the running time of the algorithm and give a tight asymptotic bound.

(b) (10 pts) Observe that we can also obtain XY as follows : XY =

$$(2^n - 2^{n/2})AC + 2^{n/2}(A+B)(C+D) + (1 - 2^{n/2})BD$$

Give a recurrence for the running time of this algorithm and give a tight asymptotic bound.

4. (30 pts) Given a directed graph G, we say that a cycle C in G is a Hamiltonian cycle if it visits each vertex exactly once. The Hamiltonian cycle problem is then the following :

   Given a directed graph G, does it contain a Hamiltonian cycle?

   (1) Is the Hamiltonian cycle problem NP? Justify.
   (2) Is the Hamiltonian cycle problem NP-hard? Justify.
   (3) Is the Hamiltonian cycle problem NP-complete? Justify.